An exact, stratified model of a meddy

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Abstract

An exact model to describe submesoscale, coherent vortices in a uniformly stratified fluid is presented. The model allows for stratification of the eddy interior, so as to agree with observations. The closed set of equations governing the evolution of the eddy on the f-plane is derived. In the case that the interior isopycnal surfaces remain horizontal the stratified analogue of the 'rodon', a special solution to the "lens equations" that govern the evolution of uniform-density, warm-core eddies, is obtained.

1 Introduction

Observations of a Meddy (Mediterranean eddy) by Armi *et al.* (1989) have revealed the following features (see Fig. 1). It consists of an anticyclonically rotating lens of salt water (angular velocity $\approx -f/3$) situated at a depth of about $1000m$. The meddy has a radial extent of approximately $25km$, a depth of about $300m$, and a lifetime of over 2 years. The density field within the meddy is stably stratified, albeit weaker than the exterior stratification. The isopycnals within the meddy typically slope in a consistent fashion and change height dramatically at the edge. Motions in the core increase linearly with increasing distance from the centre.

Meddies are one particular class of submesoscale, coherent vortices, observations and models of which hae been reviewed in McWilliams (1985). In particular, he introduced a simple model of a steady circular vortex that may be stratified in the interior. The model is underdetermined — there is one more unknown than there are equations — and one is free to prescribe an "eddy-like", monopolar pressure field, from which the azimuthal velocity and stratification follow. The boundary of the eddy in this model is, however, ill-defined. Other models assume the interior of the eddy to have either constant density (Gill, 1981; Dugan *et al*, 1982; Ruddick, 1987), or the same density gradient as the exterior (Zhmur and Pankratov, 1990, Meacham, 1992). Gill's (1981) study determines the shape and exterior velocity structure of a (basically 2D) elliptical eddy based on quasigeostrophy and hydrostacy assuming the potential vorticity to be constant in the exterior. This approach has been extended by Zhmur and Pankratov (1990) and Meacham (1992) by considering 3D ellipsoidal regions with different but uniform potential vorticity in the interior and by matching interior and exterior solutions. In Ruddick's (1987) study the eddy is residing at the interface of two infinitely deep and therefore motionless layers. Attention was consequently concentrated solely on the interior dynamics, assuming somewhat unrealistically that the velocities in the exterior, stratified region vanish. In our model a similar approach is taken, except that the exterior fields are instead considered to be unresolved (and solvable by, for example, the approach taken by Zhmur and Pankratov, 1990).

To describe the observed interior stratification a simple, exact model of an ellipsoidal, stratified eddy in a rotating stratified sea is proposed below. In this model, the eddy is enclosed by a surface of vanishing perturbation pressure, and the velocity and density fields are linear, and perturbation pressure field quadratic functions of the spatial coordinates. These have time-dependent coefficients whose time evolution is determined by a closed set of ordinary differential equations, that can be solved explicitly in particular circumstances.

2 Exact stratified eddy model

Let us consider the inviscid Navier-Stokes equations on the f-plane, scaled with 'external' scales: reference density ρ_0, Coriolis parameter f and reduced gravity $g' = g\epsilon$, where g denotes the acceleration of gravity and ϵ the scale of the overall density perturbation relative to ρ_0. Regular perturbation expansion in ϵ leads, in lowest order, to the following dimensionless equations for a Boussinesq fluid:

$$\frac{Du}{Dt} - v = -\frac{\partial p}{\partial x}, \tag{1a}$$

$$\frac{Dv}{Dt} + u = -\frac{\partial p}{\partial y}, \tag{1b}$$

$$\frac{Dw}{Dt} = -(\frac{\partial p}{\partial z} + \rho), \tag{1c}$$

where D/Dt denotes the material derivative. Because both particle and phase speeds of disturbances are much smaller than the speed of sound, and also because the vertical scales of motion are much smaller than the scale height of the ocean (which exceeds its depth), the ocean is an incompressible fluid:

$$\nabla \cdot \mathbf{u} = 0. \tag{1d}$$

and hence

$$\frac{D\rho}{Dt} = 0, \tag{1e}$$

Here u, v, w are the velocity components along x, y, z directions in a Cartesian frame of reference whose origin is located at the center of the eddy; ρ and p are the density and pressure fields expanded about the uniform and linearly varying reference state, respectively. The eddy is considered to exist within an enclosed region exterior of which the fluid is assumed to be linearly stratified: $\rho_e(z) = -zN^2/f^2$, to which the exterior pressure field $p_e(z)$ is hydrostatically related. Here N denotes the Brunt-Väisälä frequency, defined as $N = -g/\rho_0 d\rho/dz$. It is useful to define perturbation pressure and density:

$$p'(\mathbf{x}, t) = p(\mathbf{x}, t) - p_e(z), \tag{2a}$$

$$\rho'(\mathbf{x}, t) = \rho(\mathbf{x}, t) - \rho_e(z), \tag{2b}$$

which are nonzero in the interior only. The edge of the eddy is enclosed by a surface on which the perturbation pressure vanishes: $p' = 0$.

While considering the motion of a homogeneous water mass in a paraboloidal basin Ball (1963) showed that the centre of gravity may execute inertial oscillations, independent of any changes in shape of the free surface. His analysis was reinterpreted in a reduced gravity context and applied to model (uniform-density) warm-core, surface eddies by Cushman-Roisin *et al.* (1985), Young (1986), Cushman-Roisin (1987) and others. It can be shown that the subsurface, stratified eddy considered presently may likewise execute inertio-buoyancy oscillations as a whole, independent of any changes in shape, orientation and size that it may exhibit. These motions of the geometric centre are here ignored however. Ball's (1963) result was based upon integral considerations. Young (1986), aiming to give a complete description of the motion of the warm-core eddy in terms of integral quantities like the centre of gravity and moments of inertia, concluded that not enough such integral relations exist. Rather, by specifying the velocity and height fields to consist of low order polynomials with time-dependent coefficients the internal structure of the eddy turns out to be describable by eight coupled ordinary differential equations, termed the lens equations by Ruddick (1987). Young (1986) solved these, up to a final quadrature; a last integration that can be accomplished in terms of elliptic integrals.

Several conserved quantities can be formulated for the equations governing a Boussinesq fluid, Eqs. (1):

1) volume V

$$V \equiv \int_D d\mathbf{x}, \tag{3a}$$

2) potential vorticity Π

$$(\omega + \hat{\mathbf{k}}) \cdot \nabla \rho,$$

where $\omega = \nabla \times \mathbf{u}$ is the vorticity and $\hat{\mathbf{k}}$ a vertical unit vector. This quantity is materially conserved and therefore, in view of the commutativity of the time-derivative and global integration operators which the Boussinesq fluid exhibits, also its integral over the eddy domain D is conserved:

$$\Pi = \int_D (\omega + \hat{\mathbf{k}}) \cdot \nabla \rho \, d\mathbf{x}. \tag{3b}$$

3) Energy E

$$E \equiv \int_D \left[\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + z\rho + p_e \right] d\mathbf{x}, \tag{3c}$$

4) the vertical component of the absolute angular momentum vector $L_a^{(z)}$

$$L_a^{(z)} = \int_D xv - yu + \frac{1}{2}(x^2 + y^2) d\mathbf{x}. \tag{3d}$$

As in Young's (1986) analysis this does not suffice to determine the complete internal evolution of the eddy, however.

Therefore we also employ a low order polynomial expansion of the velocity, density and pressure field:

$$u_i(\mathbf{x}, t) = u_{ij}(t) x_i x_j, \tag{4a}$$

$$\rho(\mathbf{x}, t) = \rho_i(t) x_i, \tag{4b}$$

$$p'(\mathbf{x}, t) = p_0(t) + \frac{1}{2} p'_{ij}(t) x_i x_j, \tag{4c}$$

where indices $i, j \in \{1, 2, 3\}$ and summation over repeated indices is implied. Matrix p'_{ij} is symmetric. Substituting these expressions with 19 unknown functions of time in (1), we obtain:

$$\frac{du_{ij}}{dt} + u_{ik} u_{kj} + \epsilon_{i3k} u_{kj} + p'_{ij} + \delta_{i3} \rho_j = 0, \tag{5a}$$

$$\frac{d\rho_i}{dt} + u_{ki} \rho_k = 0, \tag{5b}$$

and

$$u_{kk} = 0, \tag{5c}$$

where ϵ_{ijk} and δ_{ij} are the anti-symmetric and Kronecker-delta tensors respectively. These constitute a total of 13 equations. The six missing follow from the boundary condition expressing that a particle once residing on the boundary remains on the boundary:

$$\frac{Dp'(\mathbf{x}, t)}{Dt} = 0 \quad at \;\; p'(\mathbf{x}, t) = 0.$$

Inserting expression $(4c)$ in both the boundary condition (employing $(4a)$ in the material derivative) as well as the description of the boundary itself leads to two polynomial expressions with seven independently varying, spatial fields that have to be satisfied simultaneously. Eliminating the spatially uniform term between these two and subsequently requiring the separate vanishing of each of the coefficients of the resulting polynomial with six spatially dependent terms leads to

$$\left(\frac{d}{dt} - \frac{1}{p_0}\frac{dp_0}{dt}\right) p'_{ij} + p'_{ik}u_{kj} + p'_{jk}u_{ki} = 0. \tag{5d}$$

This leaves us with a closed set of 19 nonlinearly-coupled, ordinary differential equations for as many unknowns, describing the evolution of an eddy in a uniformly stratified, rotating medium having a different internal stratification.

No notion of applying the model to oceanic eddies (except the arguments to validate the Boussinesq approximation) has been introduced up to now. The model equally applies to laboratory eddies. However, if we intend to apply the model to oceanic eddies, the observed disparity in horizontal and vertical scales L and H has to be brought into the description. In such a case it is usefull to rescale the horizontal and vertical scales and velocities separately and Eqs. (5) change only by the appearance of the square of the aspect ratio $a = H/L << 1$ in front of the acceleration terms in the vertical momentum equation. For $i = 3$ Eq. (5a) simplifies to:

$$p'_{3j} = -\rho'_j,$$

where perturbation density $\rho' = \rho'_i x_i$. As the interior is less stratified than the exterior $\rho'_3 > 0$ (see Fig. 1(c)) (Dugan *et al.* 1982, Armi *et al.* 1989). The exterior density field is in this case given by $\rho_e(z) = -Sz$, with $S = (NH)^2/(fL)^2$ denoting the Burger number, which implies

$$\rho_i = \rho'_i - S\delta_{3i} \tag{6}$$

3 Special cases

3.1 Disk-shaped eddy

A solution of (5) is given by the disc shaped perturbation pressure field

$$p' = \frac{1}{2}\rho'_3(H^2 - z^2). + \frac{1}{2}\Omega(\Omega+1)(x^2+y^2),$$

where $u_{21} = -u_{12} \equiv \Omega$ and the central pressure p_0 has been determined by assuming that the vertical scale H of the eddy is known. If also horizontal scale L and interior stratification ρ'_3 are given, the angular velocity Ω can be obtained from

$$\Omega(\Omega+1) = -\rho'_3\frac{H^2}{L^2}$$

It sahould be recalled that $\rho'_3 > 0$. As ρ'_3 is a sizeable part of N^2/f^2 the right-hand side can be expressed as a fraction of S. The vortex is necessarily a high pressure, anticyclonic eddy with $\Omega \in (-1, 0)$.

3.2 Stratified rodon

Let us consider the case that isopycnal surfaces stay flat: $\rho_1 = \rho_2 = 0$. From (5b) this implies $u_{31} = u_{32} = 0$ and $u_{33} = -1/\rho_3 d\rho_3/dt$; i.e. horizontal uniformity of the vertical velocity field. With the hydrostatic assumption, equation (5d) for p'_{13} and p'_{23} implies that the horizontal velocities have no shear in the vertical $u_{13} = u_{23} = 0$. Consequently, again from ($5d$), $p'_{33} = c_0 p_0 \rho_3^2$, where c_0 is a constant. However, because of hydrostacy, $p'_{33} = -\rho'_3$, and

$$\frac{1}{p_0}\frac{dp_0}{dt} = -\frac{1}{\rho_3}\frac{d\rho_3}{dt}\left(2 - \frac{\rho_3}{\rho'_3}\right).$$

The right-hand side reduces to $-(u_{11} - u_{22})$ only when the Burger number vanishes, in which case the equations for $u_{11}, u_{12}, u_{21}, u_{22}, p'_{11}, p'_{12}, p'_{22}$ and p_0 formally become identical to the lens equations (see Cushman-Roisin *et al* 1985, 1987, Young, 1986, Cushman-Roisin, 1987, and Ruddick, 1987). This leads to some inconsistencies, however, as $0 < \rho'_3 < S$, whereas for static stability $\rho_3 < 0$.

A useful particular solution of (5), valid even for arbitrary aspect ratio and Burger number, is the stratified analogue of the rodon (Cushman-Roisin *et al* 1985). This is, in the present context, an anticyclonic, steadily rotating ellipsoid of fiex shape in which the horizontal divergence vanishes ($u_{11} + u_{22} = 0$ and hence $u_{33} = 0$). The isopycnal field is horizontal and, as the central pressure, is constant in time. Motions are purely horizontal, lacking vertical shear.

4 Discussion and Conclusions

A complete set of equations, (5), describing the evolution of an interiorly stratified eddy in a uniformly stratified, rotating medium has been derived. The set has been shown to reduce formally to the lens equations in the limit that the aspect ratio and Burger number are small. For this set, formerly derived to describe constant density, warm-core eddies, solutions have been obtained, which show that the eddy executes simultaneous shape, size and orientation changes that occur superinertial, inertial and either sub- or superinertial respectively (Young, 1986). In the present application, where the surroundings is uniformly stratified, inertial and superinertial oscillations are subject to radiative damping. Hence, it is likely that only orientation changes will remain. The case of small Burger number has some limitations, however. It is satisfactory therefore that a stratified analogue of the rodon, which rotatees steadily at subinertial frequencies, satisfies the equations of motion unconditionally.

More detailed analysis of the equations needs to be performed. In particular the case with sloping isopycnal surfaces needs to be addressed as this feature is suggested in the observations of Armi *et al*, 1989 (see Fig. 1(b)). Also the adjustment of the exterior in response to the revolving meddy needs more carefulk consideration.

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